It is possible to calculate average Nusselt number as

$$\overline{Nu} = \frac{\bar{h}L}{k} = 2A\sqrt{Re_L} \tag{2.69}$$

Let us revisit the thermal conditions for the wall again.We consider the case where:

$$T_w = T_\infty + Cx^\lambda$$

i.e.,

$$T_w - T_\infty = Cx^\lambda \tag{2.70}$$

C and λ being constants. The heat transfer rate at the wall is as before given by

$$q_w^{''} = -k\frac{\partial T}{\partial y}|_{y=0} \tag{2.71}$$

$$q_w^{''} = -k(T_w - T_\infty)\frac{\partial \theta}{\partial \eta}|_{\eta=0}\frac{\partial \eta}{\partial y}$$

$$q_w^{''} = -kC\ x^\lambda\sqrt{\frac{U_\infty}{\nu x}}\frac{\partial \theta}{\partial \eta}|_{\eta=0}$$

$$q_w^{''} = \frac{-kC\sqrt{U_\infty}}{\sqrt{\nu}}\ \theta^{'}(0)x^{\lambda-1/2} \tag{2.72}$$

For the constant heat flux along the wall, the expression should be independent of x or in other words, we can say

$$\lambda = \frac{1}{2} \tag{2.73}$$

2.5 APPROXIMATE METHODS FOR FLAT PLATE BOUNDARY LAYER

Flow over a heated flat plate for different Prandtl numbers $\left(\frac{\delta}{\delta_T} \sim (Pr)^{1/3}$ for $Pr > 1$ or $\frac{\delta}{\delta_T} \sim (Pr)^{1/2}$ for $Pr < 1\right)$ has been illustrated in Fig. 2.4. Governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.74}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{2.75}$$

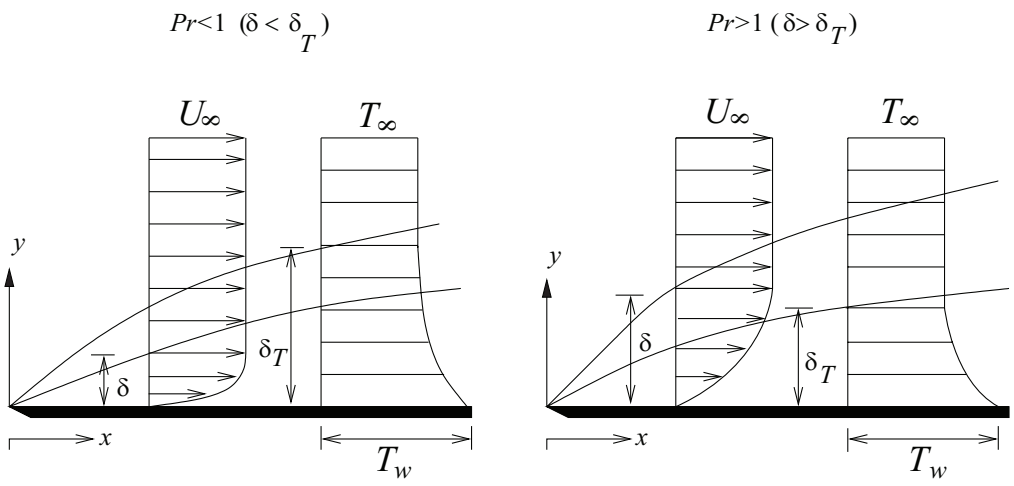

Figure 2.4 Velocity and thermal boundary layers (a) for $\delta < \delta_T$ (b) for $\delta > \delta_T$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2} \tag{2.76}$$

The boundary conditions are:

$$at\ y = 0, u = 0 = v, T = T_w$$
$$at\ y = \delta,\ u = U_\infty,\ at\ y = \delta_T, T = T_\infty$$

Integral method due to von Karman and Pohlhausen can be used to integrate Eq. (2.75)

$$\underbrace{\int_0^\delta u\frac{\partial u}{\partial x}dy}_{\text{I}} + \underbrace{\int_0^\delta v\frac{\partial u}{\partial y}dy}_{\text{II}} = \underbrace{\int_0^\delta -\frac{1}{\rho}\frac{dp}{dx}dy}_{\text{III}} + \underbrace{\int_0^\delta \nu\frac{\partial^2 u}{\partial y^2}dy}_{\text{IV}} \tag{2.77}$$

Term III is zero for flow over flat plate (because $dp/dx = 0$). Term II may be written as

$$[vu]_0^\delta - \int_0^\delta \frac{\partial v}{\partial y}udy = \left[vu|_\delta - vu|_0\right] + \int_0^\delta u\frac{\partial u}{\partial x}dy \quad \text{(from continuity)}$$

From continuity equation, one can write

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

or

$$\int_0^\delta \frac{\partial v}{\partial y}dy = -\int_0^\delta \frac{\partial u}{\partial x}dy$$

or

$$v|_{y=\delta} = -\int_0^\delta \frac{\partial u}{\partial x}dy$$

Term II finally becomes

$$-U_\infty \int_0^\delta \frac{\partial u}{\partial x} dy + \int_0^\delta u \frac{\partial u}{\partial x} dy$$

Equation (2.77) can be written as

$$2\int_0^\delta u\frac{\partial u}{\partial x}dy - U_\infty \int_0^\delta \frac{\partial u}{\partial x}dy = -\nu \frac{\partial u}{\partial y}|_{y=0}$$

or

$$\frac{d}{dx}\int_0^\delta u^2 dy - U_\infty \frac{d}{dx}\int_0^\delta u dy = -\nu \frac{\partial u}{\partial y}|_{y=0} \tag{2.78}$$

Equation (2.78) is the momentum integral equation for flow over a flat plate. The terms on the left-hand side are to be noted. In these terms, differentiation with respect to x, and integration with respect to y is interchanged as the upper limit of integration is independent of x. Momentum integral equation for a non-zero pressure gradient surface will be discussed later.

Let us assume the velocity profile as

$$u = C_0 + C_1 y + C_2 y^2 + C_3 y^3$$
$$\frac{u}{U_\infty} = C_0 + C_1\eta + C_2\eta^2 + C_3\eta^3, \quad \text{where } \eta = y/\delta$$

We already know that η is called similarity parameter; despite the growth of boundary layer in x direction, u/U_∞ remains similar for same $\frac{y}{\delta}$ at any x.

Boundary conditions:
Application of the condition,

$$at \;\; y = 0, u = 0, \;\; \text{or, } at \;\; \eta = 0, \frac{u}{U_\infty} = 0, \;\; \text{makes } \; C_0 = 0$$

Application of the condition,

$$at \;\; y = 0, \frac{\partial^2 u}{\partial y^2} = 0 \;\; \left(\text{comes from } \; u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}\right),$$
$$\text{or, } at \;\; \eta = 0, \frac{\partial^2 (u/U_\infty)}{\partial \eta^2} = 0,$$
$$\text{or, } [2C_2 + 6C_3\eta]_{\text{at } \; \eta=0} = 0 \;\; \text{yields } \; C_2 = 0$$

Application of the condition,

$$at \;\; y = \delta, u = U_\infty, \;\; \text{or, } at \;\; \eta = 1, \frac{u}{U_\infty} = 1$$
$$\text{or, } \left[C_1\eta + C_3\eta^3\right]_{\eta=1} = 1, \;\; \text{makes, } \; C_1 + C_3 = 1$$

$$at \;\; y=\delta, \frac{\partial u}{\partial y}=0 \;\; \text{or, } at \;\; \eta=1, \frac{\partial}{\partial \eta}\left[\frac{u}{U_\infty}\right]=0$$
$$\text{or, } [C_1+3C_3\eta^2]_{\eta=1}=0, \;\; yields \;\; C_1+3C_3=0$$

Finally we get

$C_1=\frac{3}{2}, C_3=-\frac{1}{2}$. The velocity profile can be written as, $\frac{u}{U_\infty}=\frac{3}{2}\eta-\frac{1}{2}\eta^3$.

By making use of the velocity profile, the first term in Eq. (2.78) can be evaluated as

$$\int_0^\delta u^2 dy=\frac{U_\infty^2}{4}\int_0^1 [9\eta^2+\eta^6-6\eta^4]\;\delta d\eta$$

$$=\frac{\delta U_\infty^2}{4}\left[\frac{9}{3}+\frac{1}{7}-\frac{6}{5}\right]=\frac{68}{35}\frac{\delta U_\infty^2}{4}$$

Evaluation of the second term of Eq. (2.78) produces

$$\int_0^\delta u dy=\frac{U_\infty}{2}\int_0^1 [3\eta-\eta^3]\;\delta d\eta=\frac{5}{4}\frac{U_\infty \delta}{2}$$

The third term of equation (2.78) $-\nu\frac{\partial u}{\partial y}|_{y=0}$ becomes

$$=-\frac{\nu}{\delta}\frac{\partial}{\partial \eta}\left[U_\infty\left(\frac{3}{2}\eta-\frac{1}{2}\eta^3\right)\right]|_{\eta=0}$$

$$=-\frac{3}{2}\frac{\nu U_\infty}{\delta}$$

Finally Eq. (2.78) can be written as

$$\frac{U_\infty^2}{4}\frac{d\delta}{dx}\left[\frac{68}{35}-\frac{5}{2}\right]=-\frac{3}{2}\frac{U_\infty \nu}{\delta}$$

which reduces to

$$\delta\frac{d\delta}{dx}=\frac{140}{13}\frac{\nu}{U_\infty}$$

On integration this gives

$$\frac{1}{2}\frac{13}{140}\delta^2=\frac{\nu x}{U_\infty}+C \tag{2.79}$$

The initial condition: $at \;\; x=0, \delta=0$ gives $C=0$. Finally,

$$\delta^2=\frac{280}{13}\frac{\nu x}{U_\infty}$$

which gives

$$\delta = 4.64\sqrt{\frac{\nu x}{U_\infty}}$$

On substituting $Re_x = \frac{U_\infty x}{\nu}$ we get

$$\delta = 4.64\frac{x}{\sqrt{Re_x}} \tag{2.80}$$

The integral form of thermal boundary layer equation may be written as

$$\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + \int_0^{\delta_T} v\frac{\partial T}{\partial y}dy = \int_0^{\delta_T} \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial y^2}dy \tag{2.81}$$

Integrating the second term by parts and applying continuity, we get

$$\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + [vT]_0^{\delta_T} - \int_0^{\delta_T} T\frac{\partial v}{\partial y}dy = \left[\frac{k}{\rho c_p}\frac{\partial T}{\partial y}\right]_0^{\delta_T}$$

or

$$\int_0^{\delta_T} u\frac{\partial T}{\partial x}dy + \int_0^{\delta_T} T\frac{\partial u}{\partial x}dy + T_\infty\, v|_{\delta_T} = -\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0}$$

where $v|_{\delta_T} = -\int_0^{\delta_T} \frac{\partial u}{\partial x}dy$ or

$$\int_0^{\delta_T} \left(u\frac{\partial T}{\partial x} + T\frac{\partial u}{\partial x}\right)dy - T_\infty \int_o^{\delta_T} \frac{\partial u}{\partial x}dy = -\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0}$$

or

$$\int_0^{\delta_T} \frac{\partial}{\partial x}[u(T - T_\infty)]dy = -\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0}$$

or

$$\frac{d}{dx}\int_0^{\delta_T} [u(T - T_\infty)]dy = -\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0} \tag{2.82}$$

This equation is called the energy integral equation. In order to solve this, let us assume the temperature profile, $\theta = \frac{T-T_w}{T_\infty - T_w} = C_0 + C_1\zeta + C_2\zeta^2 + C_3\zeta^2$ where $\zeta = \frac{y}{\delta_T}$ is a similarity parameter.

Applying thermal boundary conditions;

$$\begin{aligned}
&\text{at} \quad y = 0; T = T_w, \text{ or} \quad \text{at} \quad \zeta = 0,\ \theta = 0 \text{ we get } C_0 = 0\\
&\text{at} \quad y = 0; \frac{\partial^2 T}{\partial y^2} = 0, \text{ or} \quad \text{at} \quad \zeta = 0, \frac{d^2\theta}{d\zeta^2} = 0 \text{ we get } \ C_2 = 0\\
&\text{at} \quad y = \delta_T; T = T_\infty, \text{ or} \quad \text{at} \quad \zeta = 1,\ \theta = 1 \text{ we get } \ C_1 + C_3 = 1\\
&\text{at} \quad y = \delta_T; \frac{\partial T}{\partial y} = 0, \text{ or} \quad \text{at} \quad \zeta = 1,\ \frac{d\theta}{d\zeta} = 0 \text{ we get } \ C_1 + 3C_3 = 0
\end{aligned}$$

Solving for C_1 and C_3 we get $C_1 = \frac{3}{2}$, and $C_3 = -\frac{1}{2}$. Therefore,

$$\frac{T - T_w}{T_\infty - T_w} = \frac{3}{2}\zeta - \frac{1}{2}\zeta^3 \tag{2.83}$$

We can also write

$$\theta - 1 \ = \frac{T - T_w - T_\infty + T_w}{T_\infty - T_w} \ = \frac{T - T_\infty}{T_\infty - T_w}$$

or

$$(T - T_\infty) \ = \ (\theta - 1)\ (T_\infty - T_w)$$

First term of the energy integral (Eq. (2.82)) can be reduced to

$$(T_\infty - T_w)\frac{d}{dx}\int_0^{\delta_T} u(\theta - 1)dy$$

$$U_\infty(T_\infty - T_w)\frac{d}{dx}\int_0^{\delta_T}\left[\frac{3}{2}\eta - \frac{1}{2}\eta^3\right]\left[\frac{3}{2}\zeta - \frac{1}{2}\zeta^3 - 1\right]dy$$

$$U_\infty(T_\infty - T_w)\frac{d}{dx}\int_0^{\delta_T}\left[\frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[\frac{3}{2}\left(\frac{y}{\delta_T}\right) - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3 - 1\right]dy$$

Substituting $\xi = \frac{\delta_T}{\delta}$, rearranging and integrating, we get

$$U_\infty(T_\infty - T_w)\frac{d}{dx}\left[\delta\left\{\frac{3}{4}\xi^2 - \frac{3}{20}\xi^4 - \frac{3}{20}\xi^2 + \frac{1}{28}\xi^4 - \frac{3}{4}\xi^2 + \frac{1}{8}\xi^4\right\}\right]$$

$$= U_\infty(T_\infty - T_w)\frac{d}{dx}\left[\delta\left\{-\frac{3}{20}\xi^2 + \frac{3}{280}\xi^4\right\}\right]$$

The RHS of energy integral equation (Eq. (2.82)) is evaluated in the following way:

$$(T - T_w) = (T_\infty - T_w)\left[\frac{3}{2}\left(\frac{y}{\delta_T}\right) - \frac{1}{2}\left(\frac{y}{\delta_T}\right)^3\right]$$

$$\frac{\partial T}{\partial y} = (T_\infty - T_w)\left[\frac{3}{2}\frac{1}{\delta_T} - \frac{3}{2}\frac{y^2}{\delta_T^3}\right]$$

or

$$-\frac{k}{\rho c_p}\frac{\partial T}{\partial y}|_{y=0} = -\alpha\frac{3}{2}\frac{(T_\infty - T_w)}{\delta_T}$$

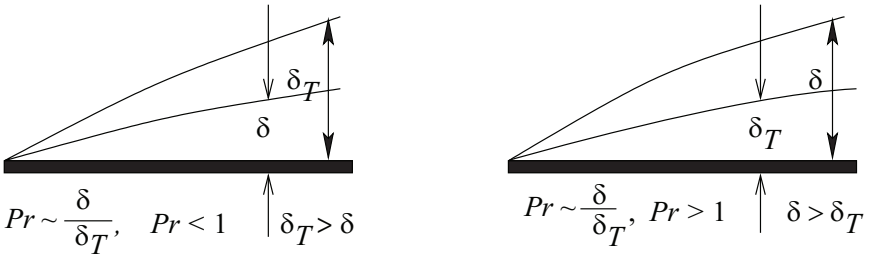

Figure 2.5 Different situations with $Pr < 1$ and $Pr > 1$

Now Eq. (2.82) becomes

$$(T_\infty - T_w)U_\infty \frac{d}{dx}\left[\delta\left\{-\frac{3}{20}\xi^2 + \frac{3}{280}\xi^4\right\}\right] = -\frac{3}{2}\frac{\alpha(T_\infty - T_w)}{\delta_T} \tag{2.84}$$

Figure 2.5 explains different situations pertaining to the growth of boundary layers for various Prandtl numbers.
For $Pr > 1,\ \delta > \delta_T,\ \xi = \frac{\delta_T}{\delta}$ becomes small, and so $\xi^4 << \xi^2$. Therefore, for $Pr > 1$ (Fig. 2.5), Eq. (2.84) becomes

$$\delta U_\infty \frac{d}{dx}\left[\delta\frac{\xi^2}{20}\right] = \frac{\alpha}{2\xi}$$

or

$$\frac{10\alpha}{U_\infty \xi} = \delta\frac{d\delta}{dx}\xi^2 + \delta^2\frac{d}{dx}\xi^2$$

where

$$\delta\frac{d\delta}{dx} = \frac{140}{13}\frac{\nu}{U_\infty},\quad and\quad \delta^2 = \frac{280}{13}\frac{\nu x}{U_\infty}\quad \text{(from Eq.(2.79))}$$

Finally, on substitution of the above two relations, Eq. (2.84) becomes

$$\xi^3 + 2x\xi\frac{d}{dx}(\xi^2) = \frac{13}{14}\frac{1}{Pr} \tag{2.85}$$

Put $\chi = \xi^3;\ \frac{d\chi}{dx} = 3\xi^2\frac{d\xi}{dx}$; substituting in Eq. (2.85)

$$\chi + \frac{4x}{3}\frac{d\chi}{dx} = \frac{13}{14}\frac{1}{Pr}$$

or

$$\frac{d\chi}{dx} + \frac{3}{4x}\chi = \frac{13}{14}\frac{1}{Pr}\frac{3}{4x}$$

This linear differential equation for which the integrating factor is evaluated as $e^{\int\frac{3}{4}\frac{dx}{x}} = e^{(3/4)ln\ x} = x^{3/4}$

$$x^{3/4}\frac{d\chi}{dx} + \frac{3}{4}\frac{\chi}{x}x^{3/4} = \frac{13\times 3}{14\times 4}\frac{1}{Pr}x^{-1/4} \tag{2.86}$$

or

$$\frac{d}{dx}(x^{3/4}\chi) = \frac{39}{56}\frac{1}{Pr}x^{-1/4} \tag{2.87}$$

$$\chi = \frac{13}{14}\frac{1}{Pr} + bx^{-3/4} \tag{2.88}$$

where b is the constant of integration.

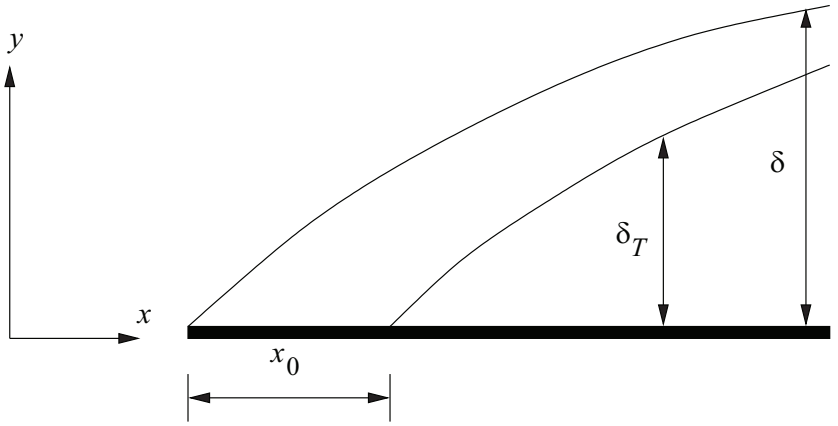

Figure 2.6 Flat plate situation with unheated initial length

Analyzing the boundary conditions as described in Fig. 2.6, we consider a short unheated length x_0 so that $at \;\; x = x_0,\; \delta_T = 0,\; \xi = 0$; and so, $at \;\; x = x_0,\; \chi = 0$. Substituting in Eq. (2.88), we get

$$b = -\frac{13}{14}\frac{1}{Pr}x_0^{3/4} \tag{2.89}$$

After substituting in Eq. (2.88) for b, we get

$$\xi^3 = \chi = \frac{13}{14}\frac{1}{Pr} - \frac{13}{14}\frac{1}{Pr}\left(\frac{x_0}{x}\right)^{3/4}$$

or

$$\frac{\delta_T}{\delta} = \xi = \frac{1}{1.026}(Pr)^{-1/3}\left[1 - \left(\frac{x_0}{x}\right)^{3/4}\right]^{1/3}$$

For $x_0 \to 0$, we finally have

$$\delta_T = \frac{\delta}{1.026}(Pr)^{-1/3} \tag{2.90}$$

Again we know that

$$h(T_w - T_\infty) = -k\frac{\partial T}{\partial y}|_{y=0}$$

On substituting for $\frac{\partial T}{\partial y}|_{y=0}$, we get

$$h = \frac{-k\frac{(T_\infty - T_w)}{\delta_T}\frac{3}{2}}{(T_w - T_\infty)}$$

or

$$h = k\frac{3}{2}\frac{1.026}{4.64x}(Pr)^{1/3}(Re_x)^{1/2}$$

(after invoking Eqs. (2.80) and (2.90)). The above expression leads to

$$\frac{hx}{k} = 0.332(Re_x)^{1/2}(Pr)^{1/3}$$

The local Nusselt number can be expressed as

$$Nu_x = 0.332(Re_x)^{1/2}(Pr)^{1/3} \tag{2.91}$$

Now the average heat transfer coefficient $\bar{h} = \int_0^L h_x dx / \int_0^L dx$ and so the average Nusselt number

$$\overline{Nu_L} = \frac{\bar{h}L}{k} = 0.664(Re_L)^{1/2}(Pr)^{1/3} \tag{2.92}$$

where $Re_L = \frac{\rho U_\infty L}{\mu}$

2.6 VISCOUS DISSIPATION EFFECTS ON LAMINAR BOUNDARY LAYER FLOW OVER FLAT PLATE

For the high speed flows, the effects due to viscous dissipation have to be considered. Now, $u = U\ f^{'}(\eta)$ and $\eta = y/\sqrt{(\nu x)/U}$, and so the the viscous dissipation term is

$$\mu\left[\left(\frac{\partial u}{\partial y}\right)^2 + ...\right]$$

$$= \mu\left[\left(\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}\right)^2 + ...\right]$$

$$= \mu\left[\left(U\ f^{''}\sqrt{\frac{U}{\nu x}}\right)^2 + ...\right]$$

$$= \mu U^2 (f^{''})^2\ \frac{U}{\nu x}$$